Graphical Linear Algebra

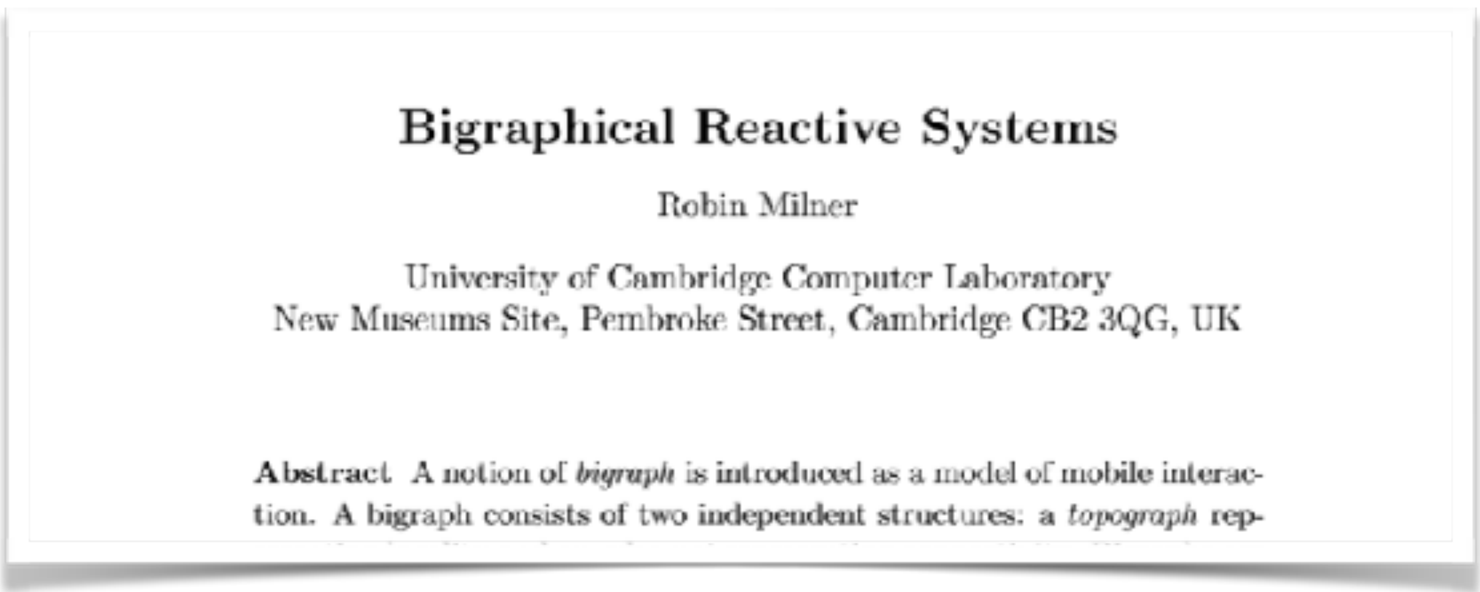
a specification language for linear algebra

Pawel Sobocinski

based on joint work with Filippo Bonchi and Fabio Zanasi

CONCUR

- My first CS conference: Concur 2001, Aalborg



Bigraphical Reactive Systems

Robin Milner

University of Cambridge Computer Laboratory
New Museums Site, Pembroke Street, Cambridge CB2 3QG, UK

Abstract A notion of *bigraph* is introduced as a model of mobile interaction. A bigraph consists of two independent structures: a *topograph* rep-

- My first talk: EXPRESS 2002, Brno (a Concur 2002, satellite workshop

Deriving Bisimulation Congruences:
A 2-categorical Approach

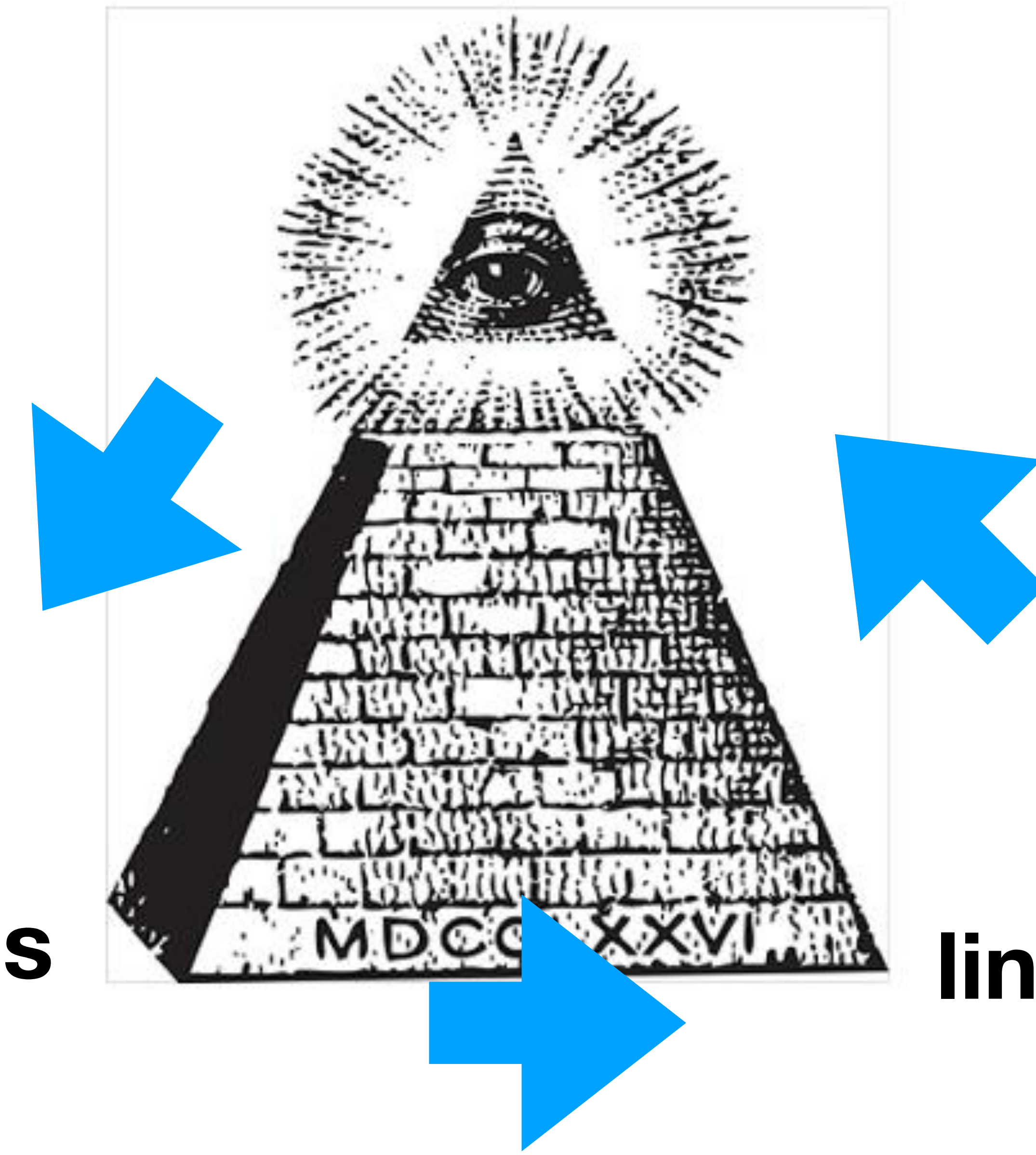
Vladimiro Sassone

COGS
University of Sussex, United Kingdom

Paweł Sobociński

BRICS
University of Aarhus, Denmark

- My first paper at Concur 2003, Marseille (with Bartek Klin)



(Jean-Raymond Abrial : how theorem provers make you treat mathematics as a branch of software engineering)

This talk: how to see linear algebra as a branch of process algebra implementations $\subseteq$ specifications

runnable, completely specified

non-deterministic, partially specified behaviour

---

functions $\subseteq$ relations

single-valued, total

non single-valued, non total

string diagrams

- showing up in a growing number of recent CS papers
  - Abramsky, Duncan, Coecke, … - Categorical Quantum Foundations and Quantum Computation
  - Mellies, … - Logic, Game Semantics
  - Ghica, Jung, … - Digital Circuits
  - Baez, Bonchi, Erbele, Fong, S., Zanasi, … - Signal Flow Graphs, Control and Systems Theory
  - Coecke, Sadrzadeh, … - Computational Linguistics
  - …

1st Workshop on String Diagrams
in Computation Logic and Physics
Jericho Tavern, Oxford
8-9 September, 2017
(satellite of FSCD,
next year satellite of CSL)

linear algebra

- the most practical mathematical theory?
  - the engine room of systems and control theory, quantum computing, network theory, …
  - mathematical physics and engineering relies on it: systems of nonlinear differential equations are solved with linear approximations
  - shows up in surprising places (Petri net invariants, PageRank is an eigenvector, SVD in data science and learning, …)
- Graphical Linear Algebra - linear algebra with string diagrams
  - focus on linear **relations** rather than on linear **maps**
  - GraphicalLinearAlgebra.net

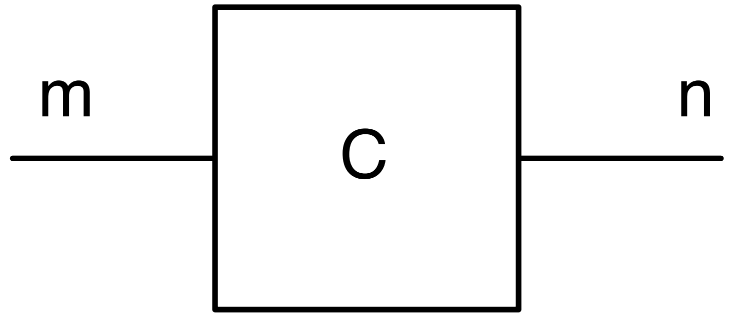
Plan

- String diagrams & diagrammatic reasoning
  - **what is it?**
  - why is it relevant for cs?
- Graphical Linear Algebra
- Fun stuff

props

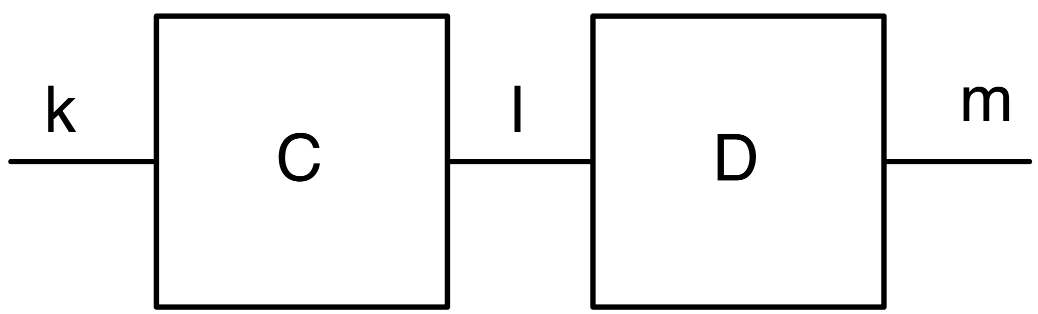
- A prop is a strict symmetric monoidal category with
  - strict means: ⊗ is associative on the nose
  - objects = natural numbers
  - $m \otimes n := m + n$ (I will usually write $m \oplus n$)
- Simple examples:
  - permutations of finite sets
  - functions between finite sets
- prop homomorphism = identity on objects symmetric monoidal functor

A string diagram



Synchronising Composition

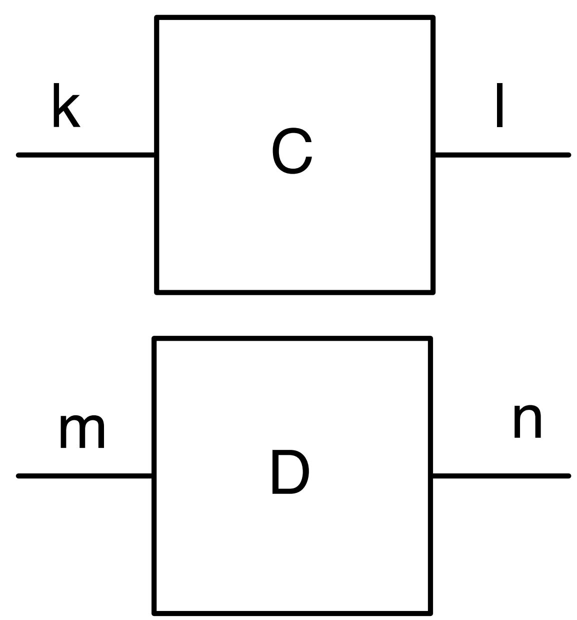
$$\frac{C : k \to l \qquad D : l \to m}{C \,;\, D : k \to m}$$



“C and D synchronise on l”

Parallel composition

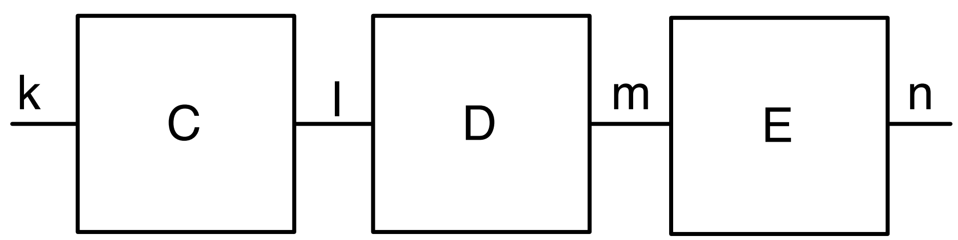
$$\frac{C : k \to l \qquad D : m \to n}{C \oplus D : k \oplus m \to l \oplus n}$$



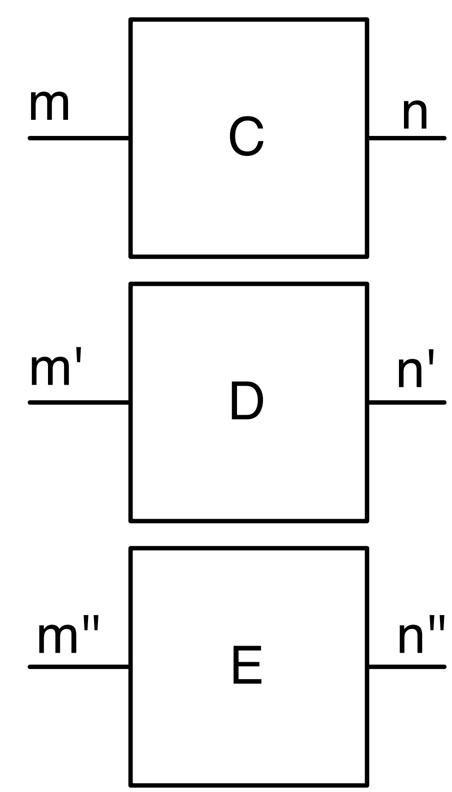
“C and D in parallel”

Perks of the notation

$$\frac{C : k \to l \qquad\qquad D : l \to m \qquad\qquad E : m \to n}{(C ; D) ; E = C ; (D ; E) : k \to n}$$

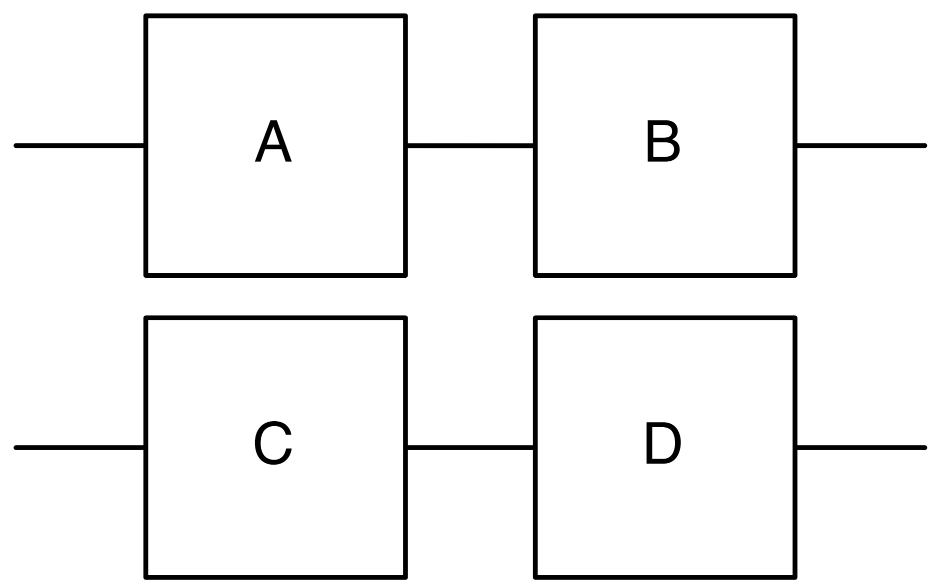


$$\frac{C : m \to n \qquad\qquad D : m' \to n' \qquad\qquad E : m'' \to n''}{(C \oplus D) \oplus E = C \oplus (D \oplus E) : \mathrm{m} \oplus \mathrm{m}' \oplus \mathrm{m}'' \to \mathrm{n} \oplus \mathrm{n}' \oplus \mathrm{n}''}$$



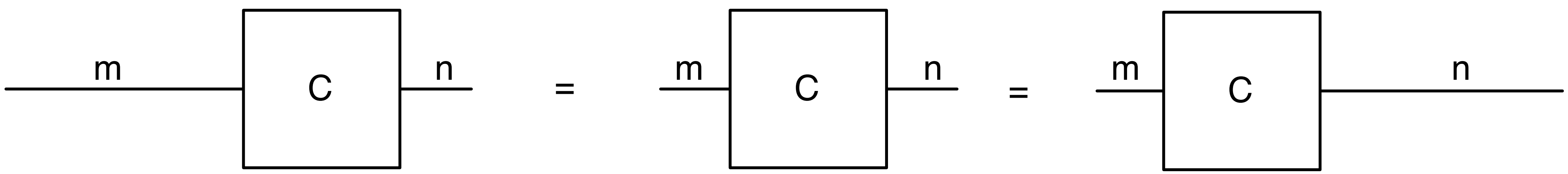
More perks

$$(A \,;\, B) \oplus (C \,;\, D) = (A \oplus C) \,;\, (B \oplus D)$$

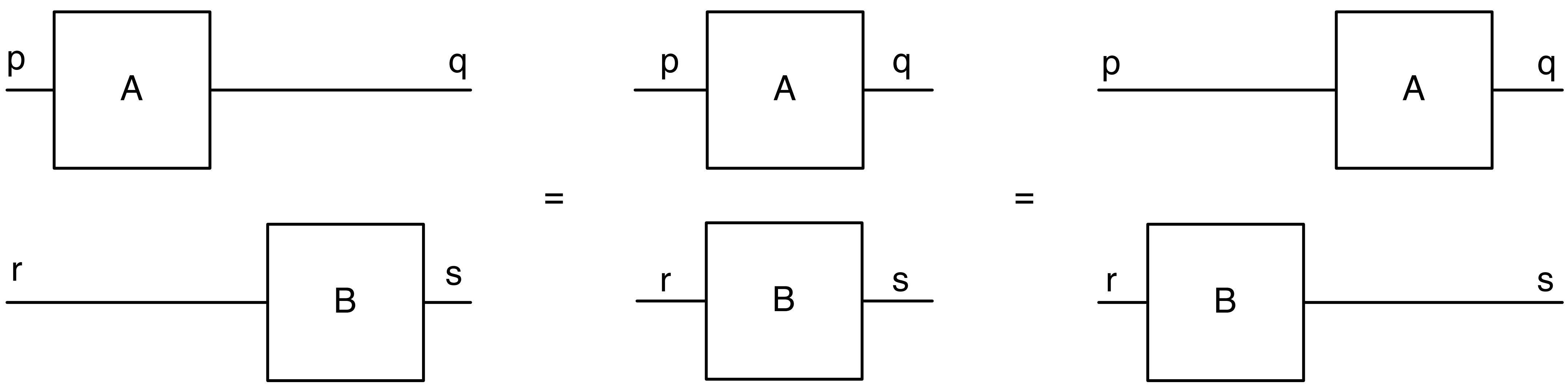


Diagrammatic reasoning

$$\frac{C : m \to n}{I_m ; C = C = C ; I_n}$$

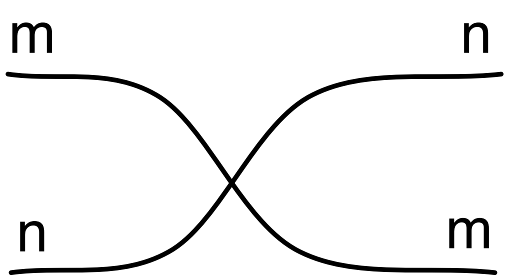


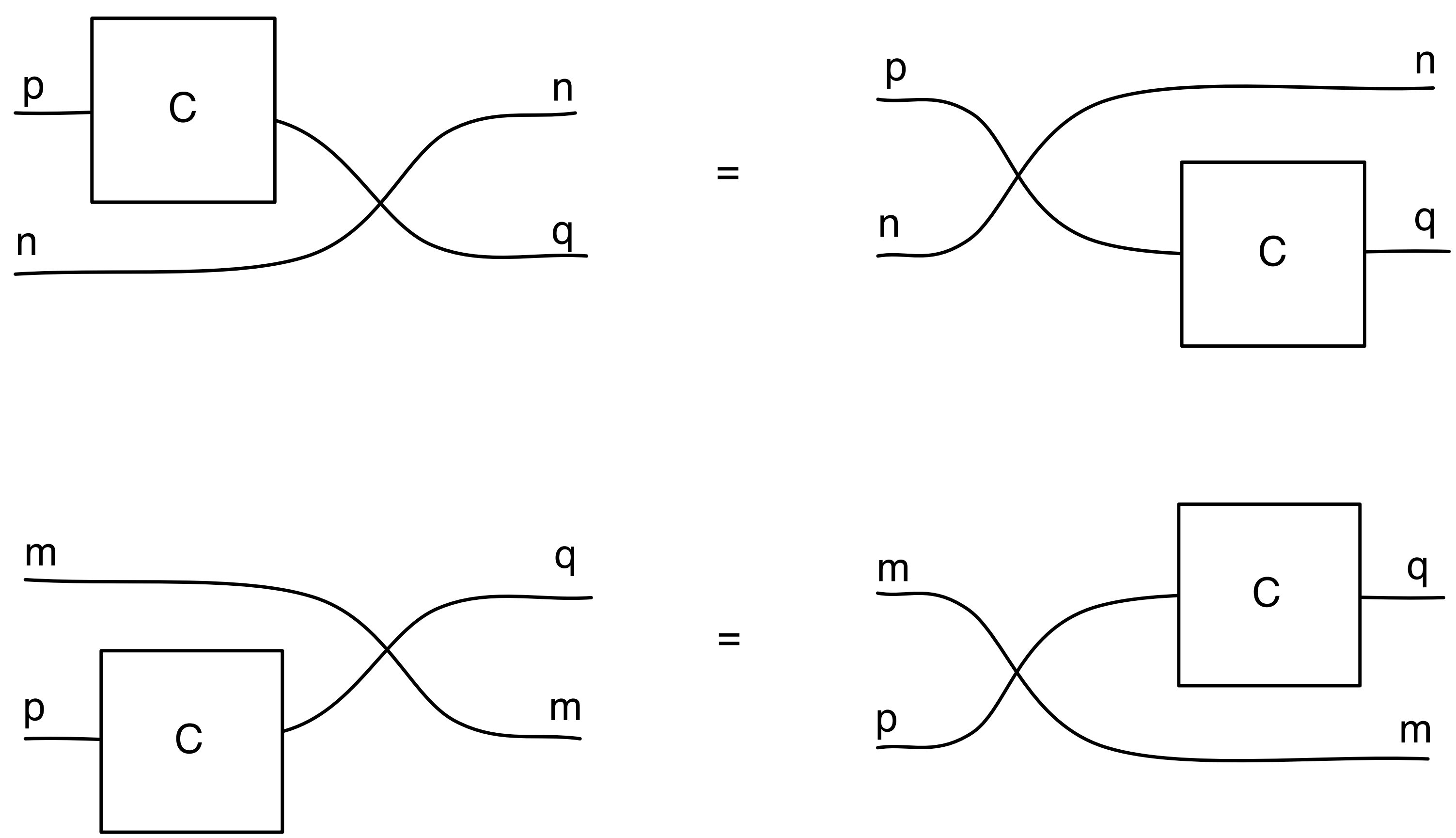
$$(A \oplus I_r) ; (I_q \oplus B) = A \oplus B = (I_p \oplus B) ; (A \oplus I_s)$$



Symmetries

$\sigma_{m,n}$: $m \oplus n \rightarrow n \oplus m$





Plan

- String diagrams & diagrammatic reasoning
    - what is it?
    - **why is it relevant for cs?**
- Graphical Linear Algebra
- Fun stuff

(commutative) monoids and groups a la 1930s universal algebra - syntax

- (presentation of) algebraic theory
  - pair $T = (\Sigma, E)$ of finite sets
- for commutative monoids:
  - signature Σ, arity: $\Sigma \to \mathbf{N}$
    - $\cdot$: 2
    - e : 0
  - equations E (pairs of **typed terms**)
    - $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
    - $x \cdot y = y \cdot x$
    - $x \cdot e = x$
- For abelian groups, additionally
  - signature: $(-)^{-1}$: 1
  - equations: $x \cdot x^{-1} = e$

(commutative) monoids and groups a la universal algebra - semantics

- To give a **model**
  - Pick carrier set X
    - $\cdot : 2$ $\Rightarrow$ $\cdot : X^2 \to X$
    - $e : 0$ $\Rightarrow$ $e : X^0 \to X$
    - $(-)^{-1} : 1$ $\Rightarrow$ $(-)^{-1} : X^1 \to X$
  - For every evaluation of variables σ: Var$\to$X, each equation must hold
- So, e.g. a **model of the algebraic theory of monoids** is the same thing as a **monoid**, in the classical sense

functorial semantics, 1960s

- Lawvere was not happy with universal algebra
  - too set theory specific
    - (e.g. topological groups morally should be a model)
  - too much ad hoc extraneous machinery
    - (e.g. countable set of variables, variable evaluation, etc.)
- Lawvere's 1963 doctoral thesis "Functorial semantics of algebraic theories" - universal algebra categorically

Lawvere theories

- Given algebraic theory (Σ,E), a category $\mathcal{L}_{(\Sigma,E)}$ with
  - objects: the natural numbers
  - arrows from m to n:
    - n-tuples of terms that (possibly) use variables $x_1, x_2, \ldots x_m$ modulo equations E
    - composition is substitution
- e.g. $2 \xrightarrow{(x_1 \cdot x_2)} 1 \qquad 2 \xrightarrow{(x_2 \cdot x_1)} 1 \qquad 1 \xrightarrow{(e,\ x_1)} 2 \xrightarrow{(x_2 \cdot x_1)} 1 \ = \ 1 \xrightarrow{(x_1 \cdot e)} 1$
- More concisely - “free category with products on the data of an algebraic theory”
- any $\mathcal{L}_{(\Sigma,E)}$ is a prop!

classical model = cartesian functor $\mathcal{L} \to$ **Set**

products in a Lawvere theory

$$
\begin{tikzcd}
m & m+n \arrow[l, "{(x_1,x_2,\ldots,x_m)}"'] \arrow[r, "{(x_{m+1},x_{m+2},\ldots,x_{m+n})}"] & n \\
 & k \arrow[ul, "{(f_1,f_2,\ldots,f_m)}"] \arrow[u, dotted, "{(f_1,\ldots,f_m,g_1,\ldots,g_n)}"'] \arrow[ur, "{(g_1,g_2,\ldots,g_n)}"'] &
\end{tikzcd}
$$

limitations of algebraic theories

- Copying and discarding **built in**

$$2 \xrightarrow{(x_1)} 1 \qquad 2 \xrightarrow{(x_2)} 1 \qquad 1 \xrightarrow{(x_1,\, x_1)} 2$$

- Consequently, there are also no bona fide operations with *coarities* other than one

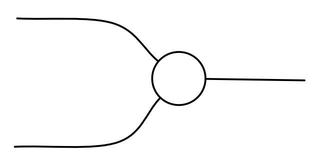
$$1 \xrightarrow{c} 2 \qquad = \qquad 1 \xrightarrow{(c_1, c_2)} 2$$

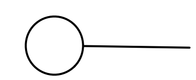
- But in quantum mechanics, computer science, and elsewhere we often need to be more careful with resources

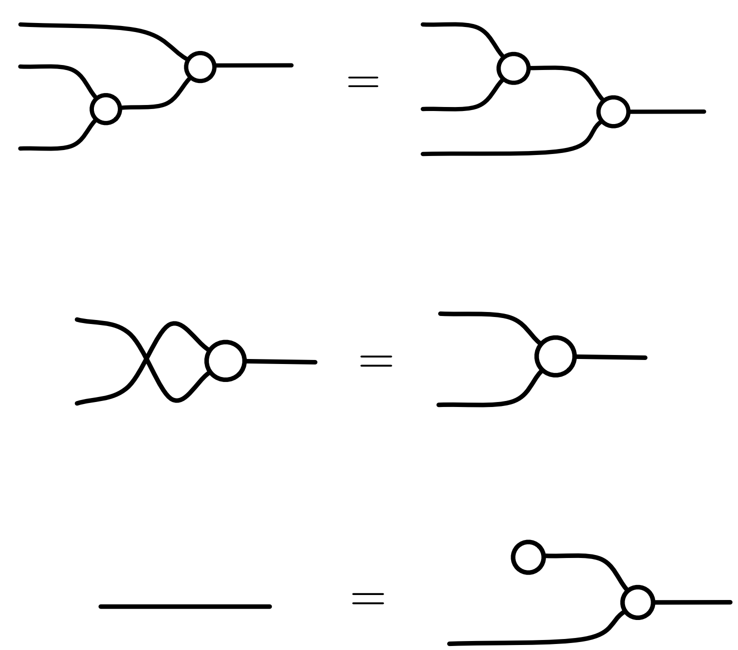
symmetric monoidal theories

- algebraic theory in the symmetric monoidal settings
- a symmetric monoidal theory is a pair of finite sets (Σ, E)
  - Σ signature, arity : $\Sigma \to N$, coarity : $\Sigma \to N$
  - E equations, pairs of **string diagrams** constructed from Σ, identity and symmetries

symmetric monoidal theory of commutative monoids

 $: (2, 1)$

 $: (0, 1)$

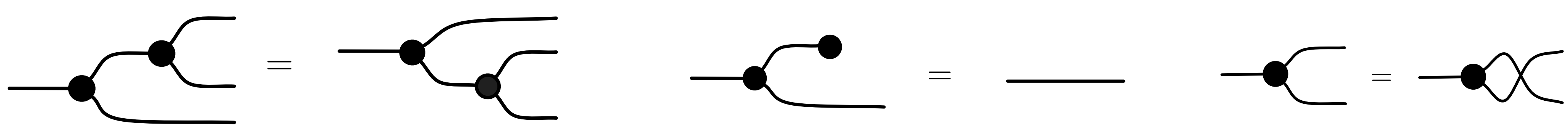


commutative monoid facts

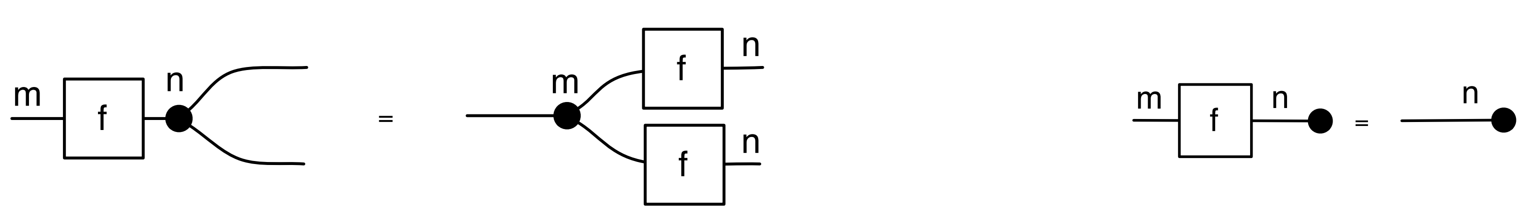
- the following are isomorphic as props
  - prop of commutative monoids
  - prop of functions between finite sets
- **not** isomorphic to the Lawvere theory of commutative monoids

folk theorem

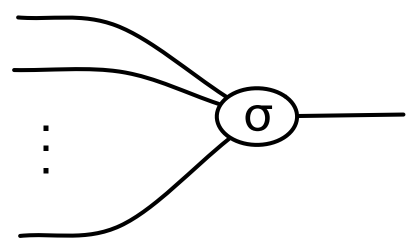
- A symmetric monoidal category **C** is cartesian iff
  - every object C∈**C** has a commutative comonoid Δ: C→C⊗C, c: C→I



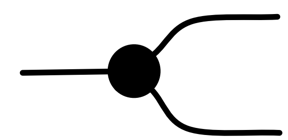
  - compatible with ⊗ in the obvious way
  - and every arrow f : m→n of **C** is a comonoid homomorphism, i.e.

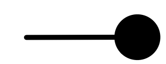


Lawvere theories as SMTs

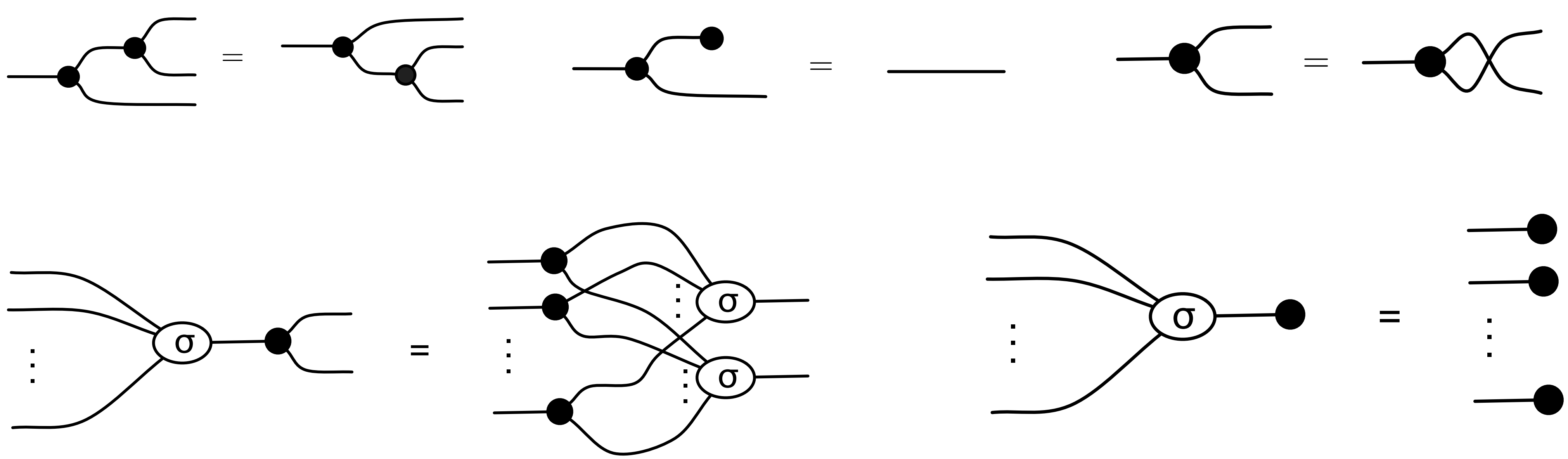


$(\sigma \in \Sigma)$

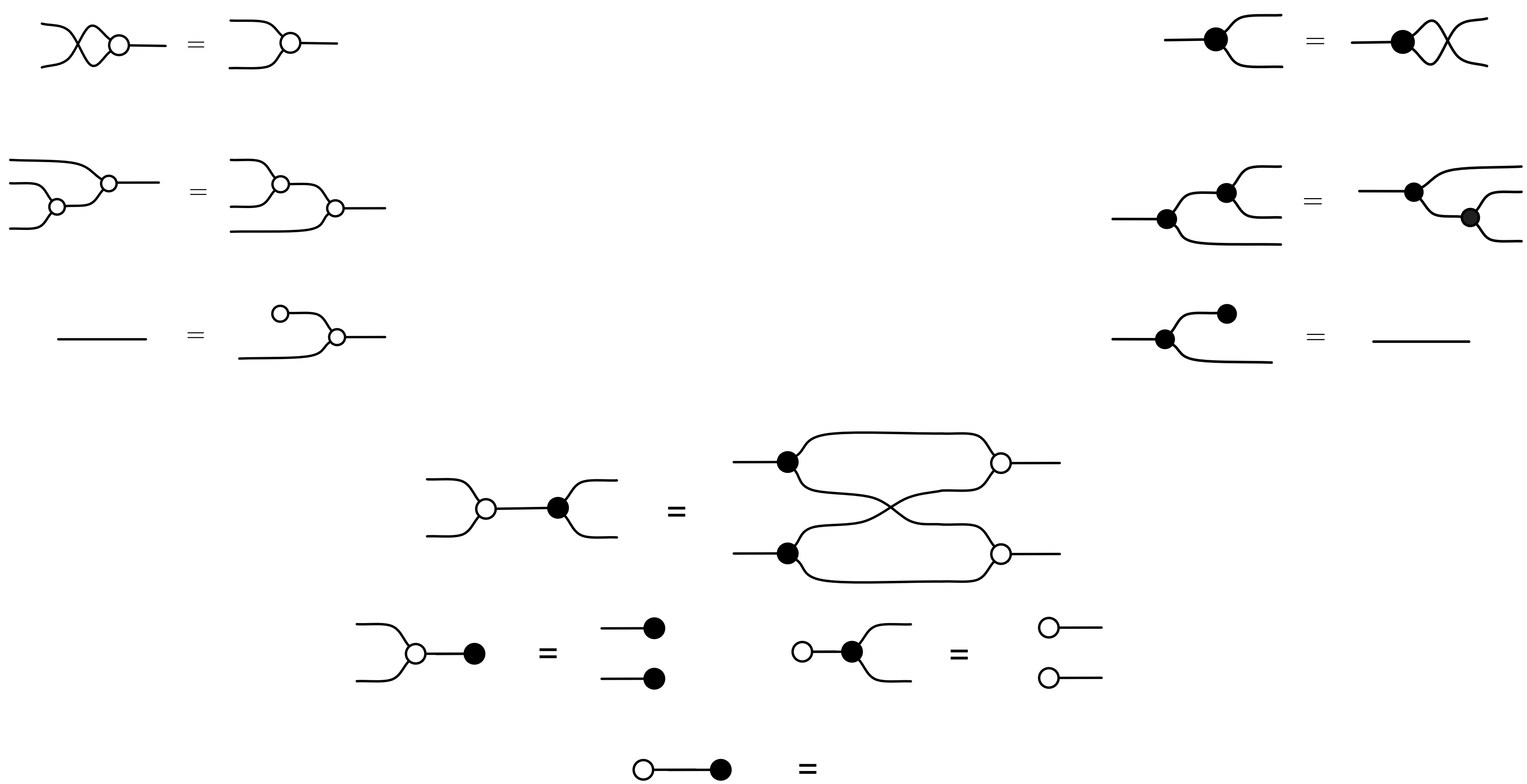




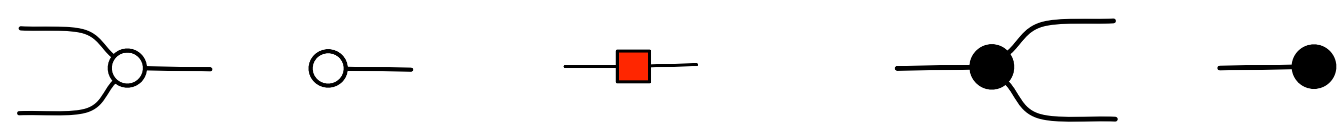
E +

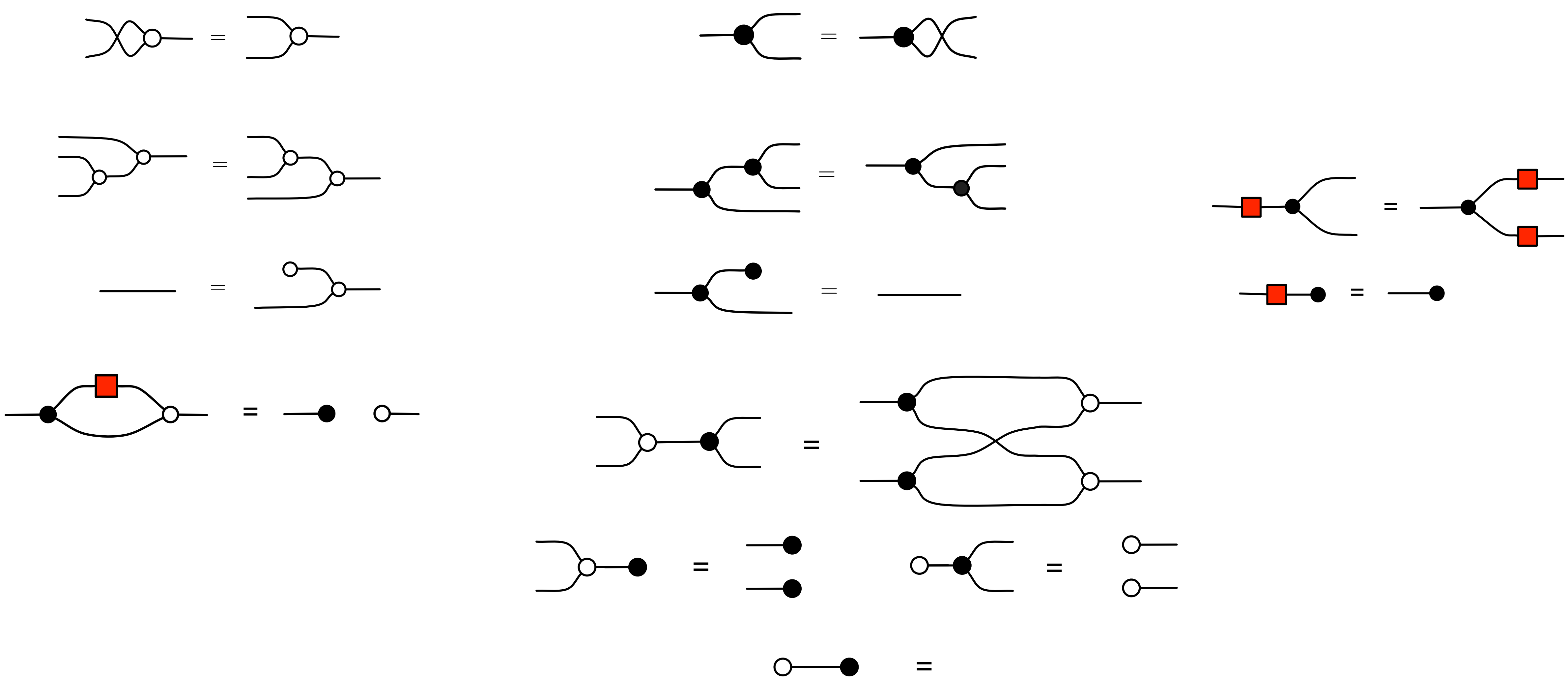


Lawvere theory of commutative monoids as SMT

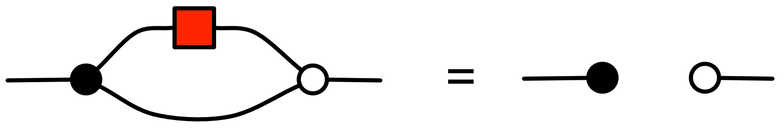


Lawvere theory of abelian groups as an SMT





- e.g. the Hopf equation



is simply the SMT version of $x \cdot x^{-1} = e$

- Lawvere theory of commutative monoids = Symmetric monoidal theory of (co)commutative bialgebra

- Lawvere theory of abelian groups = Symmetric monoidal theory of (co)commutative Hopf algebras

So bialgebras and Hopf algebras are, respectively, monoids and groups in a resource sensitive universe.

Plan

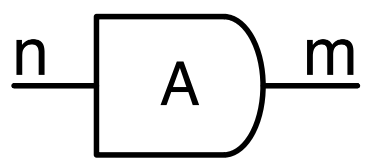
- String diagrams & diagrammatic reasoning
  - what is it?
  - why is it relevant for cs?
- **Graphical Linear Algebra**
- Fun stuff

Linear relation

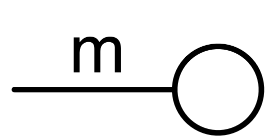
- Definition. Suppose V, W are k-vector spaces. A linear relation R from V to W is a linear subspace of $V \times W$
  - i.e.
  - $(0_V, 0_W) \in R$
  - if $(v,w), (v',w') \in R$ then $(v+v', w+w') \in R$
  - if $(v,w) \in R$ and $\lambda \in k$ then $(\lambda v, \lambda w) \in R$

Why linear relations?

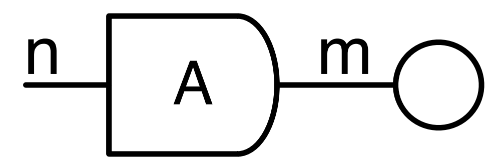
- any $m \times n$ matrix A gives lin. relation $\{ (\mathbf{x}, A\mathbf{x}) \mid x \in k^n \} \subseteq k^n \times k^m$



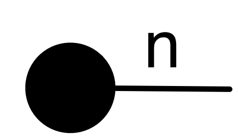
- the singleton $(\mathbf{0}, *)$ is a linear relation $\subseteq k^m \times k^0$



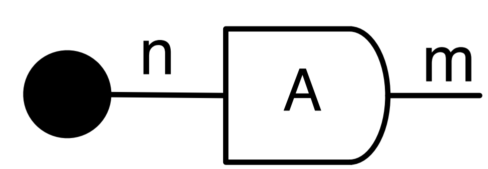
- composing gives the kernel of A



- the set $\{ (*, \mathbf{x}) \mid \mathbf{x} \in k^n \}$ is a linear relation $\subseteq k^0 \times k^n$



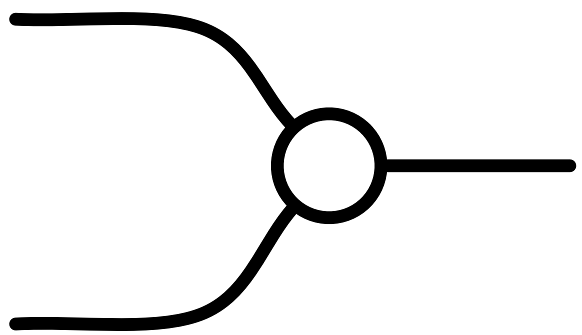
- composing gives the image of A



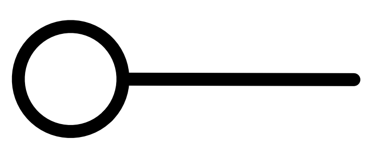
Graphical linear algebra

String diagrammatic syntax for linear relations with a sound and fully complete axiomatisation called **Interacting Hopf Algebra**

The signature, pt 1



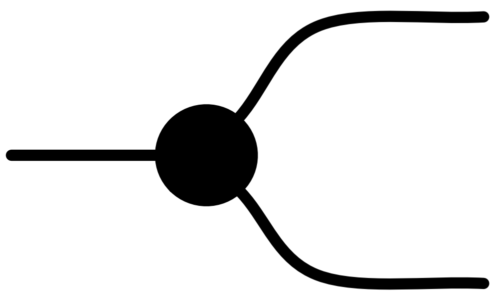
$$\Leftrightarrow \left\{ \left(\begin{pmatrix} x \\ y \end{pmatrix}, x + y \right\} \subseteq k^2 \times k$$

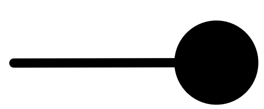


$$\Leftrightarrow \{*, 0\} \subseteq k^0 \times k^1$$

+ mirror images

The signature, pt 2

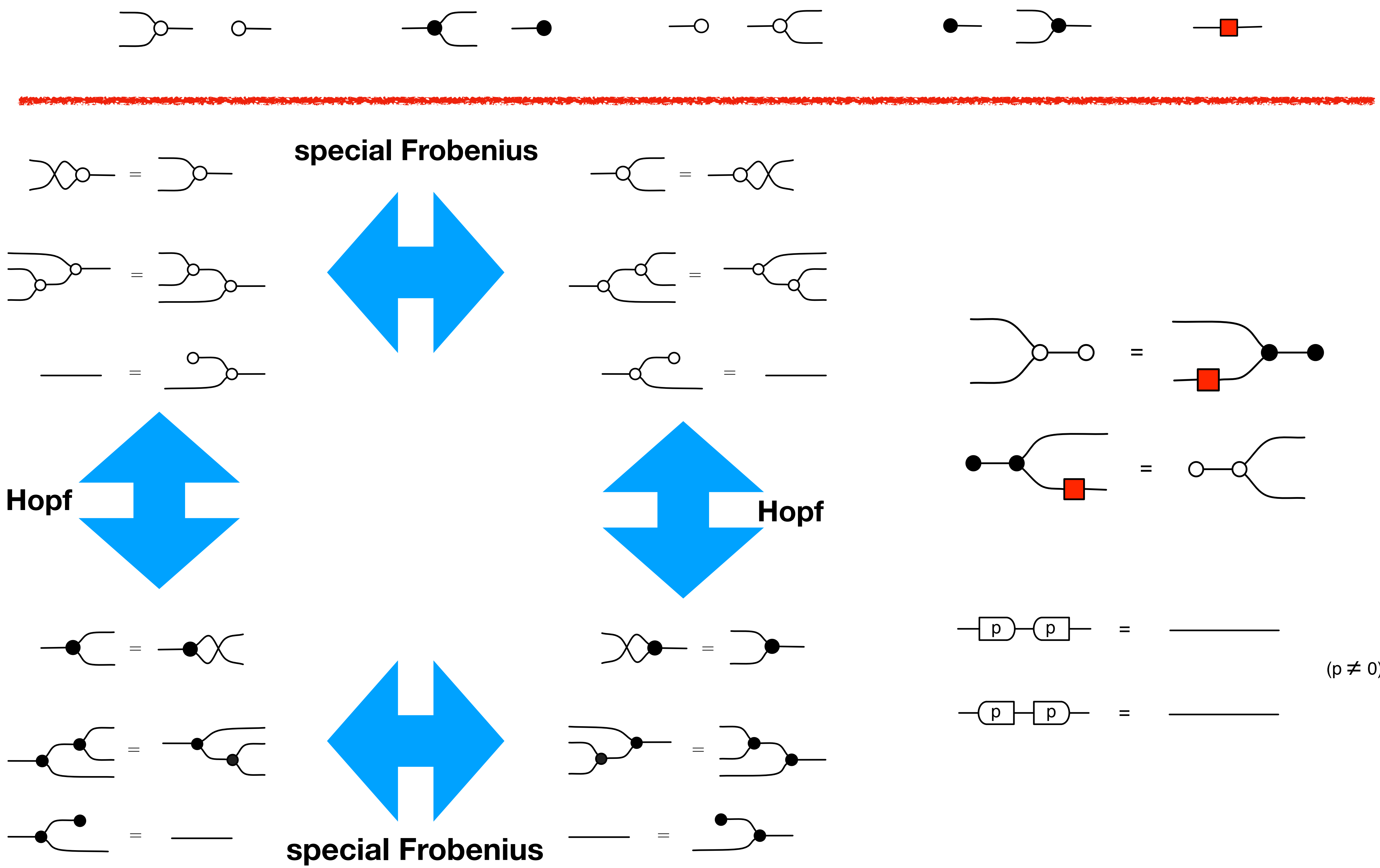
 $\Leftrightarrow$ $$\left\{ x, \begin{pmatrix} x \\ x \end{pmatrix} \right\} \subseteq k \times k^2$$

 $\Leftrightarrow$ $$\{x, *\} \subseteq k \times k^0$$

+ mirror images

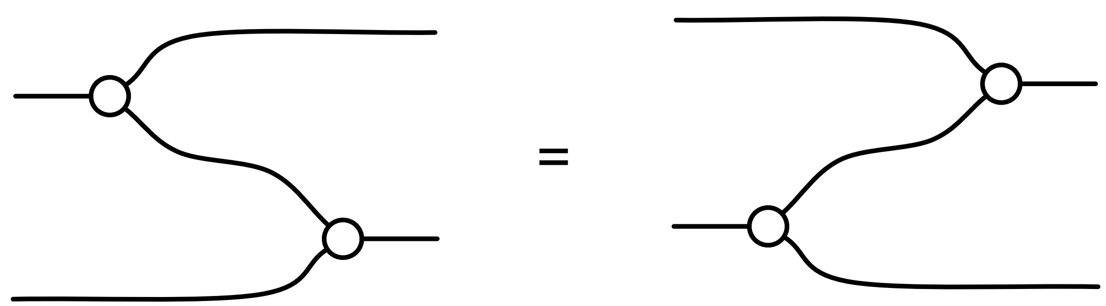
interacting Hopf algebras

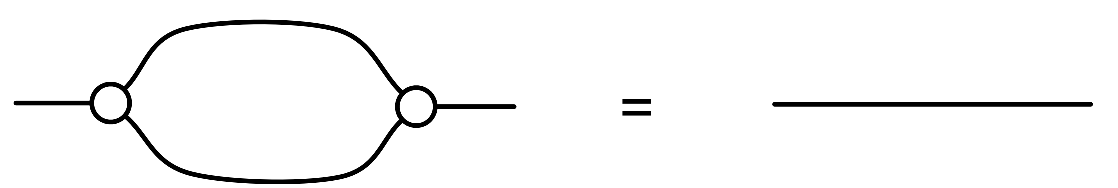
Bonchi, S., Zanasi, JPAA 2017



cf. Coecke, Duncan. Interacting quantum observables, NJP 2011

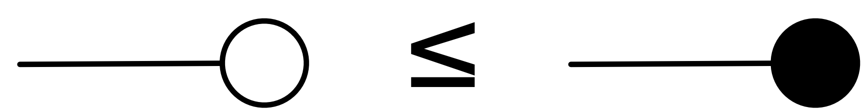
(special) Frobenius monoids





Theorem

IH ≅ LinRel



extends this to on iso of 2-categories

Bonchi, Holland, Pavlovic, S. Refinement for signal flow graphs, CONCUR'17

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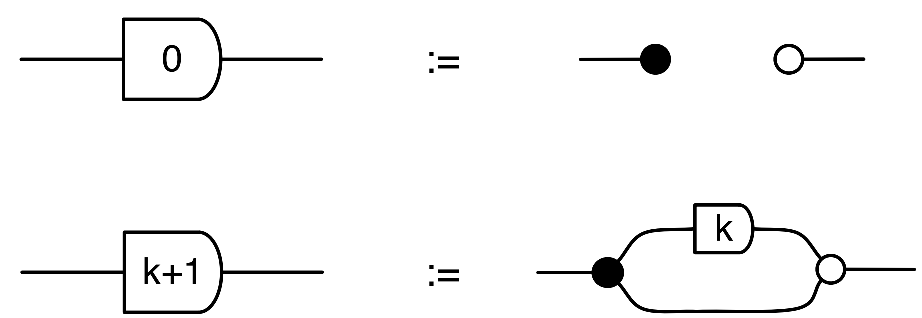
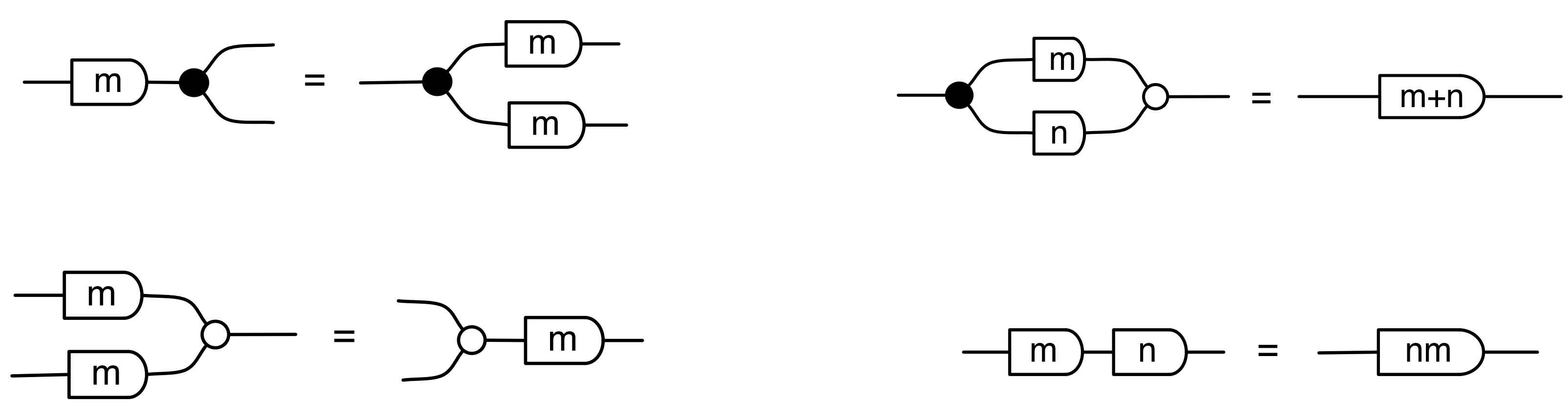
naturals as string diagrams

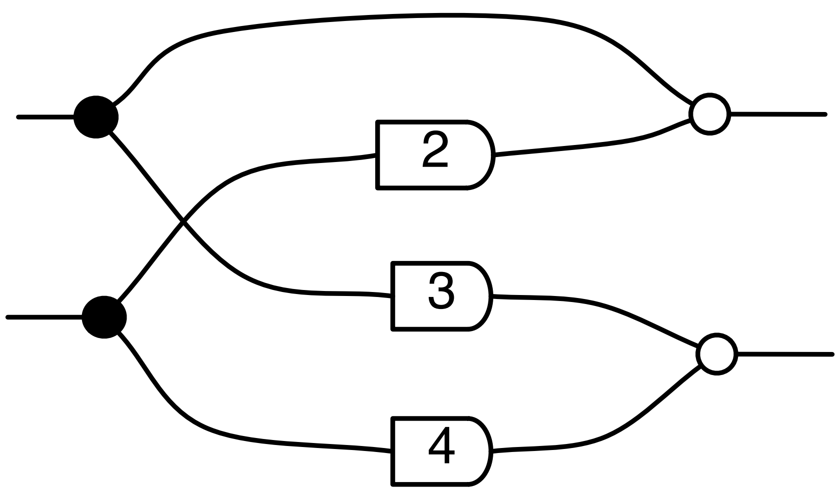
- naturals as *syntactic sugar*



- some easy lemmas



correspondence with matrices

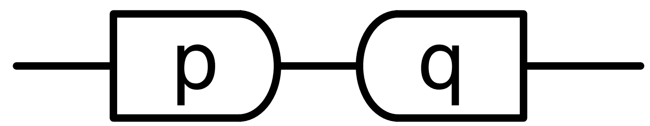


$$\Longleftrightarrow \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

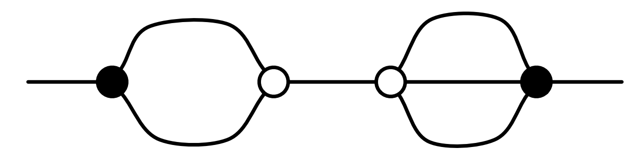
- in general, the ijth entry is the *number of paths* from the jth port on the left to the ith port on the right

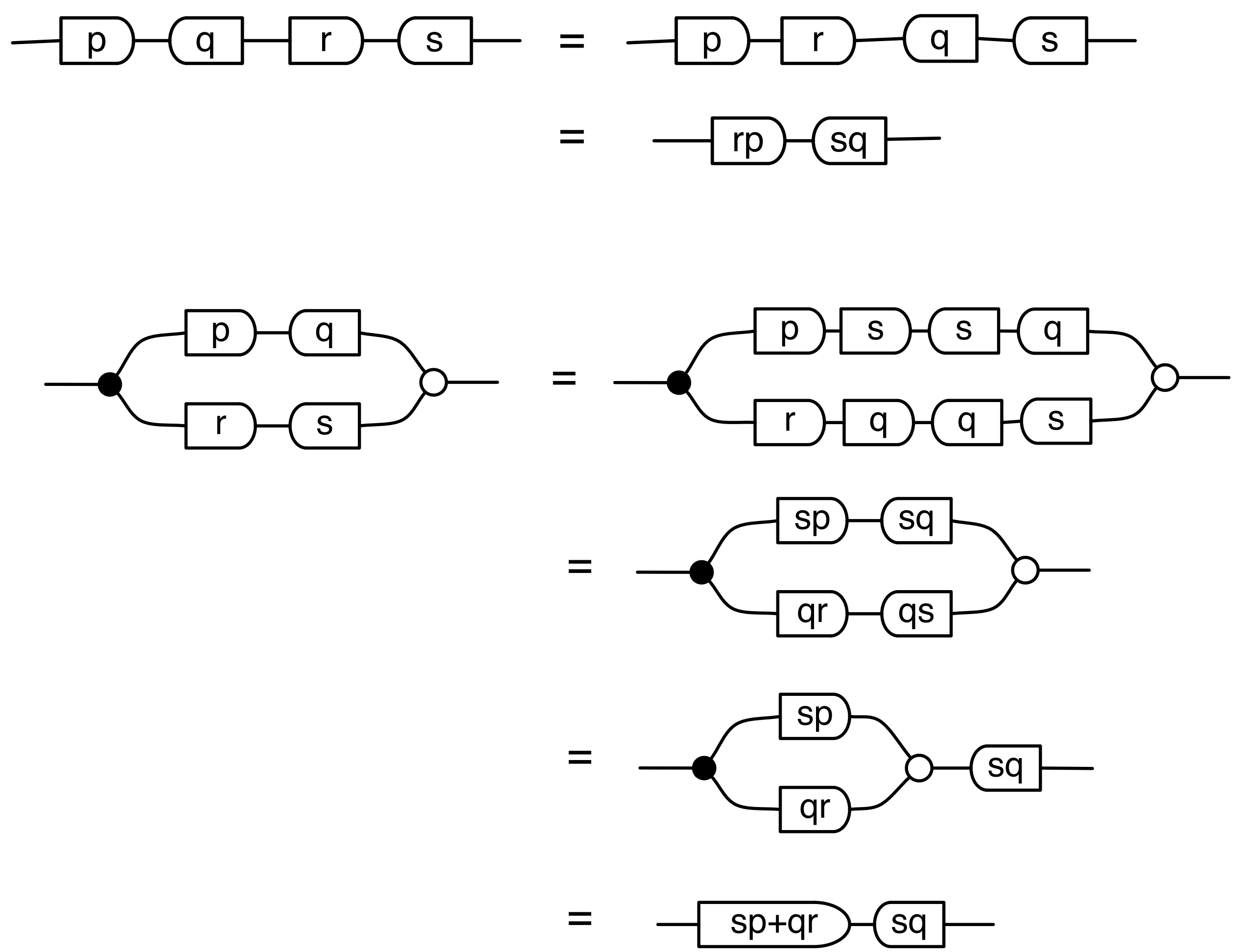
rational numbers

p/q :=



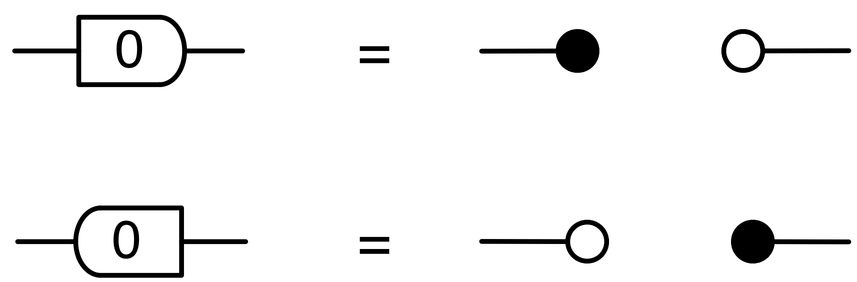
e.g. 2/3 is



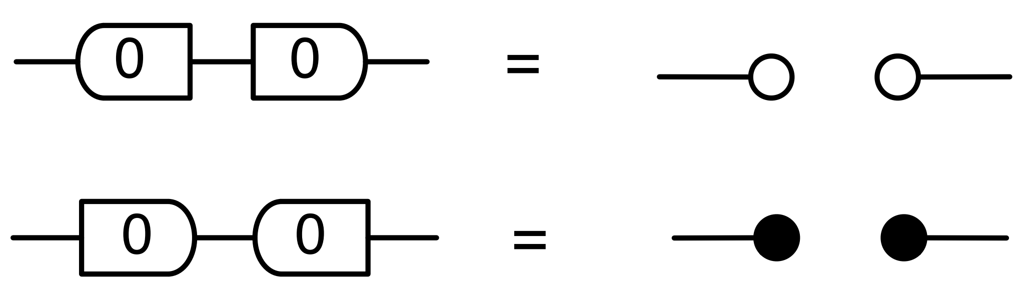


division by 0

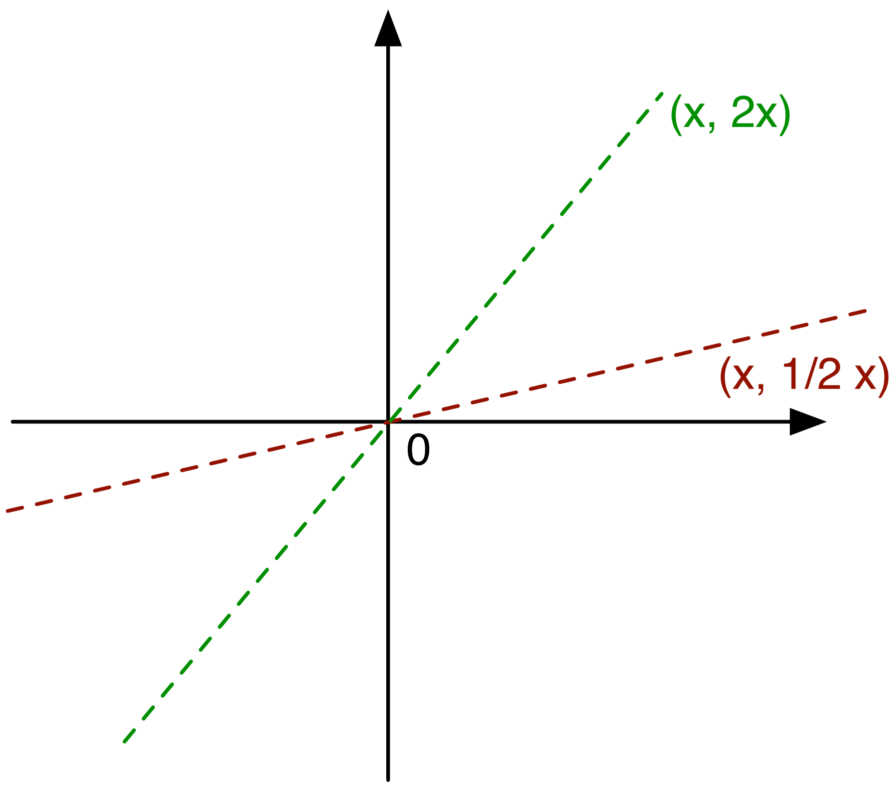
(fixing the deficiencies of the usual syntax)



Two ways of interpreting “0/0”



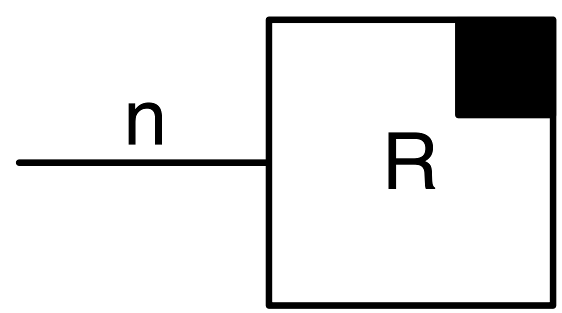
projective arithmetic ++



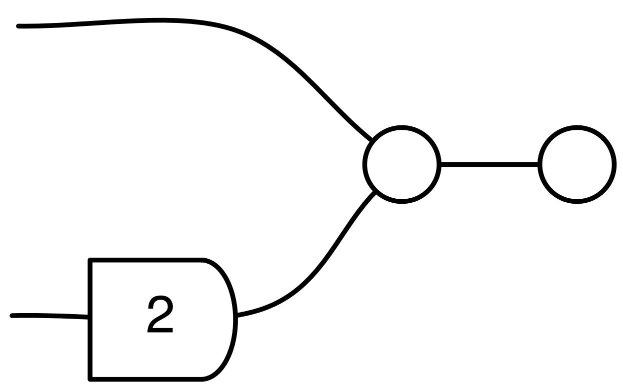
- projective arithmetic identifies rationals with 1-dim spaces (lines) of $\mathbf{Q}^2$
    - p -> { (x,px) | x ∈ **Q** }
    - ∞ : { (0, x) | x ∈ **Q** }
- The extended system includes all the subspaces of $\mathbf{Q}^2$, in particular:
    - the unique zero dimensional space { (0, 0) }
    - the unique two dimensional space { (x,y) | x,y ∈ **Q** }

Linear subspaces

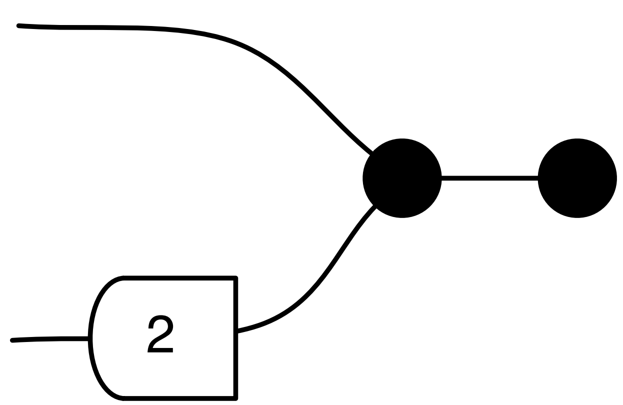
- **Observation.** Linear subspaces of kn are in 1-1 correspondence with string diagrams



Some examples

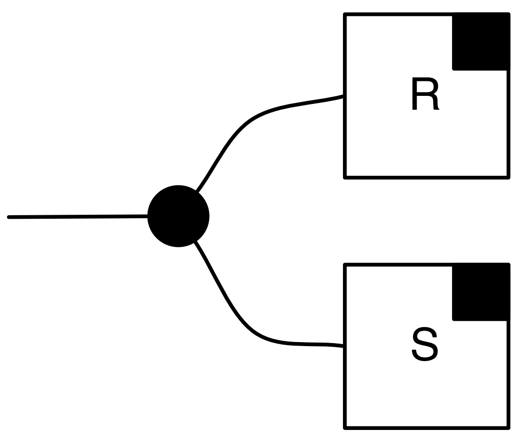


$$\left\{ \left(\begin{pmatrix} x \\ y \end{pmatrix}, * \right) \mid x + 2y = 0 \right\} \subseteq k^2 \times k^0$$

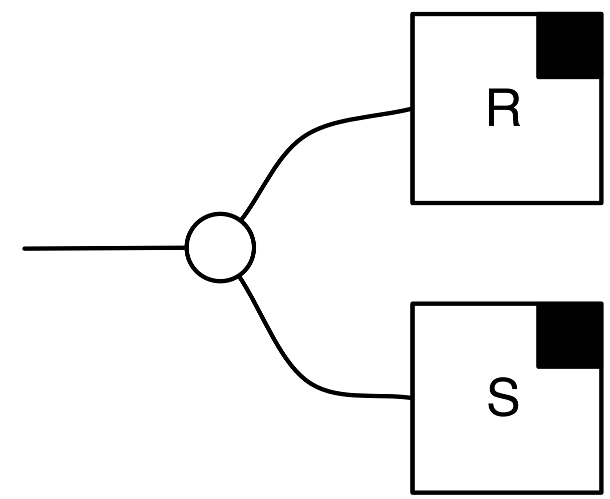


$$\left\{ \left(a \begin{pmatrix} 1 \\ 2 \end{pmatrix}, * \right) \mid a \in k \right\} \subseteq k^2 \times k^0$$

Intersection and sum of spaces

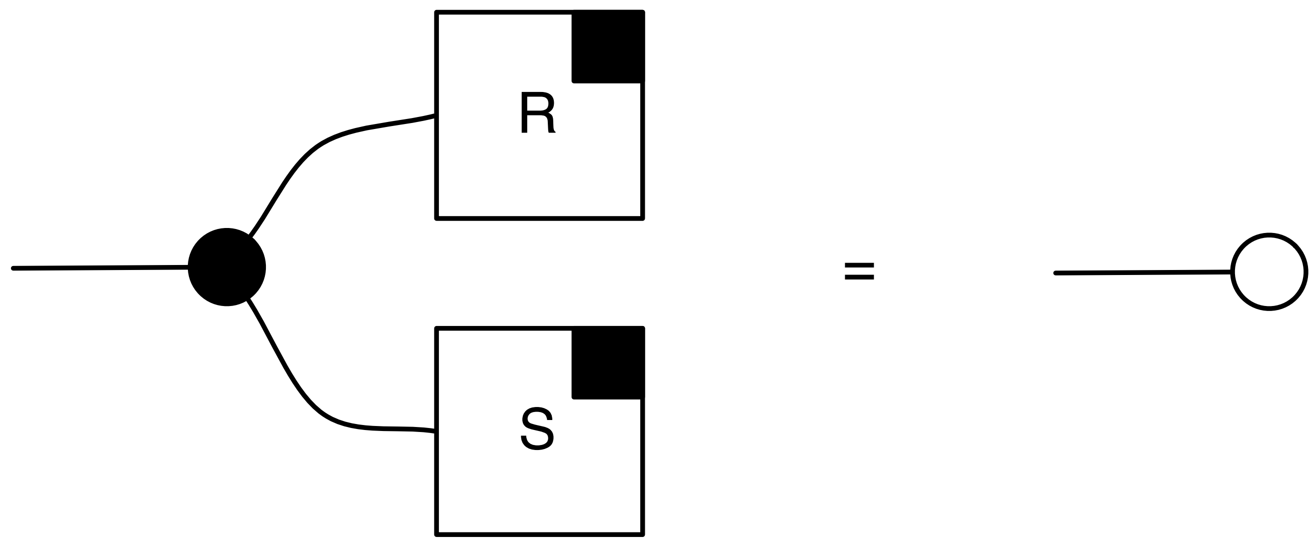


intersection

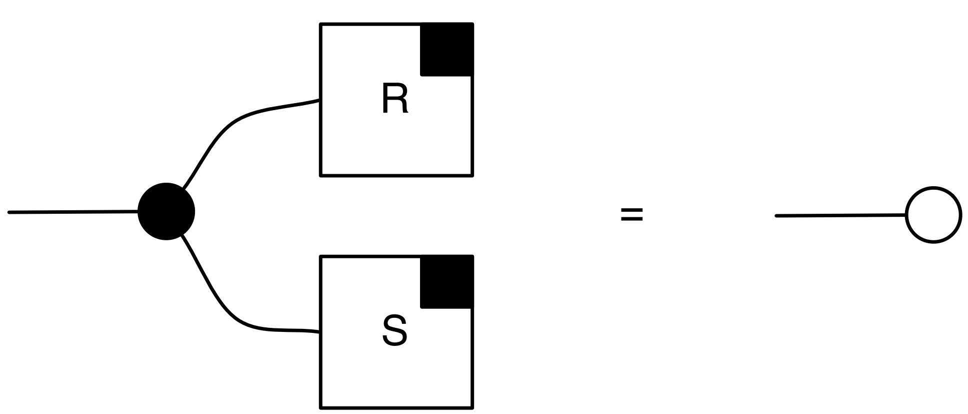


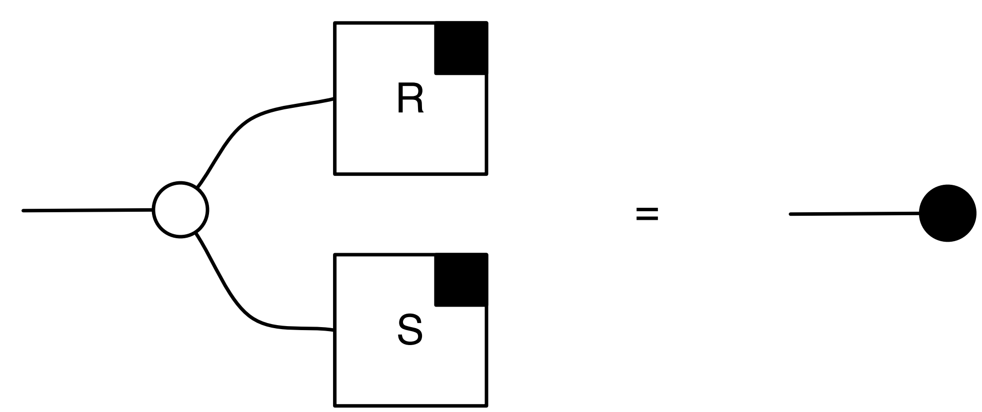
sum

linear independence



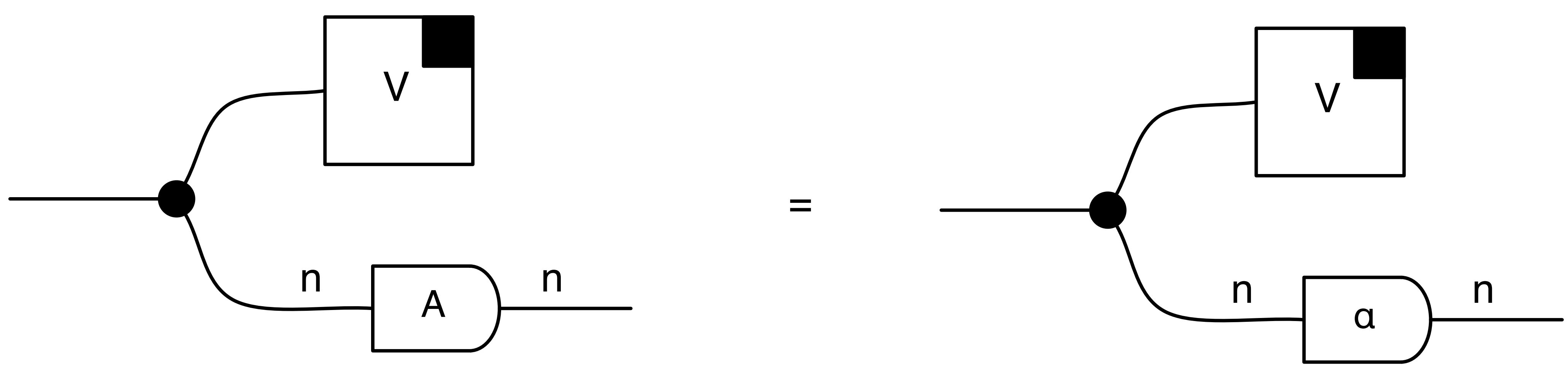
decomposition into linearly independent subspaces R and S





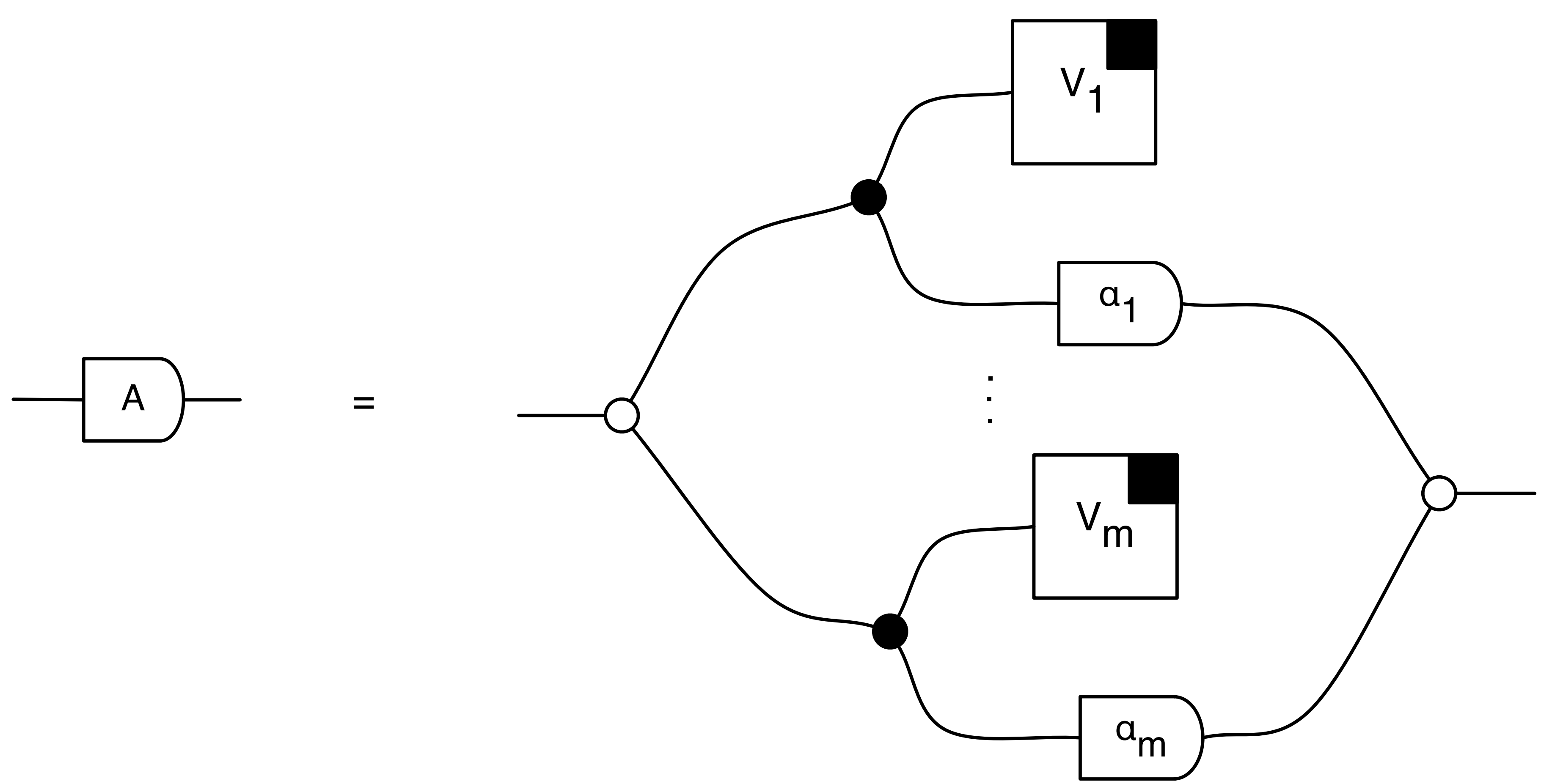
Eigenvalues & eigenspaces

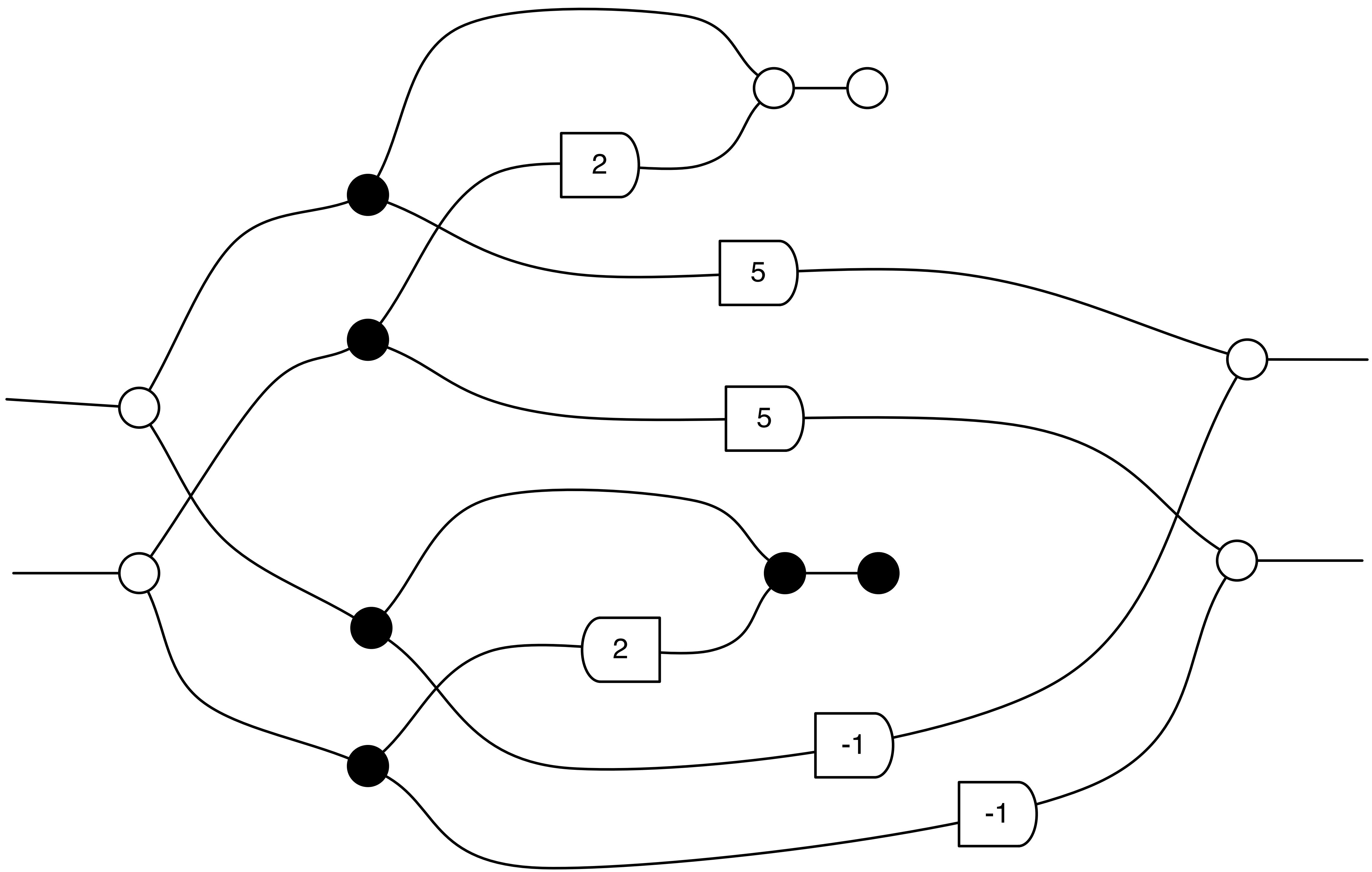
- V is an *eigenspace* of A: $k^n \to k^n$ with *eigenvalue* $a \in k$ when:

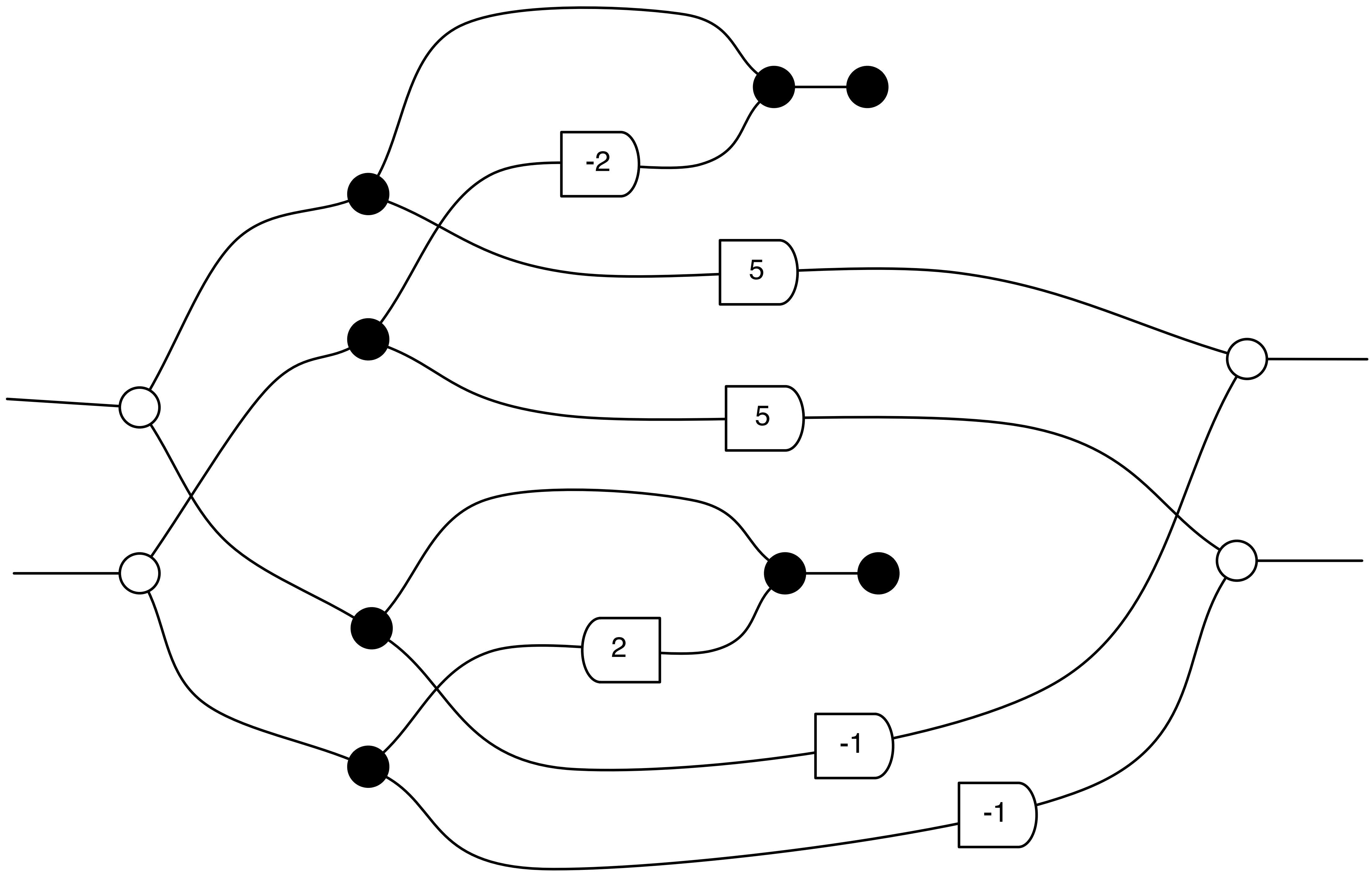


Spectral decomposition

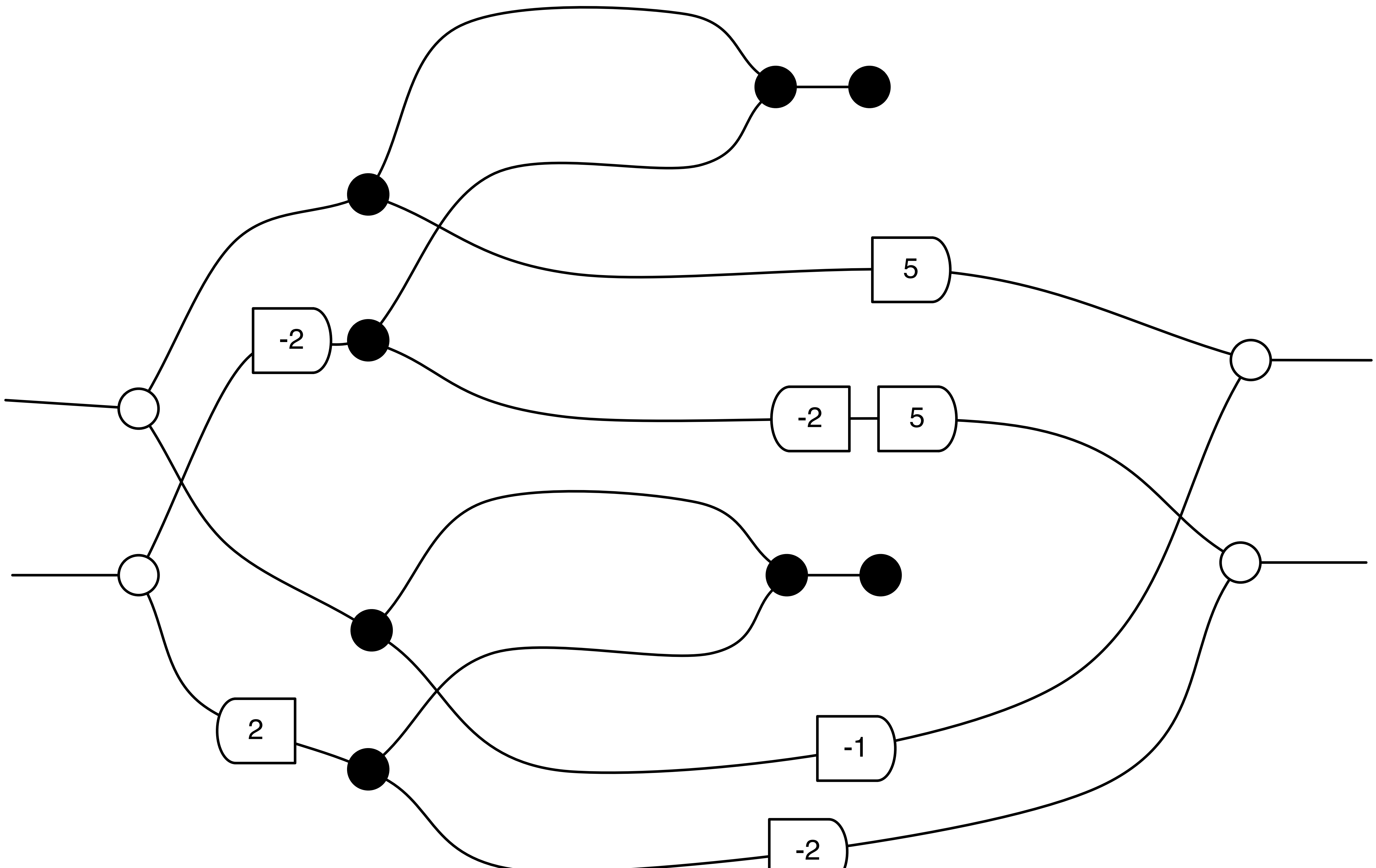
- A has a spectral decomposition when we can find a decomposition of k^n into eigenspaces $V_1, V_2, \ldots, V_m$ and eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_m$



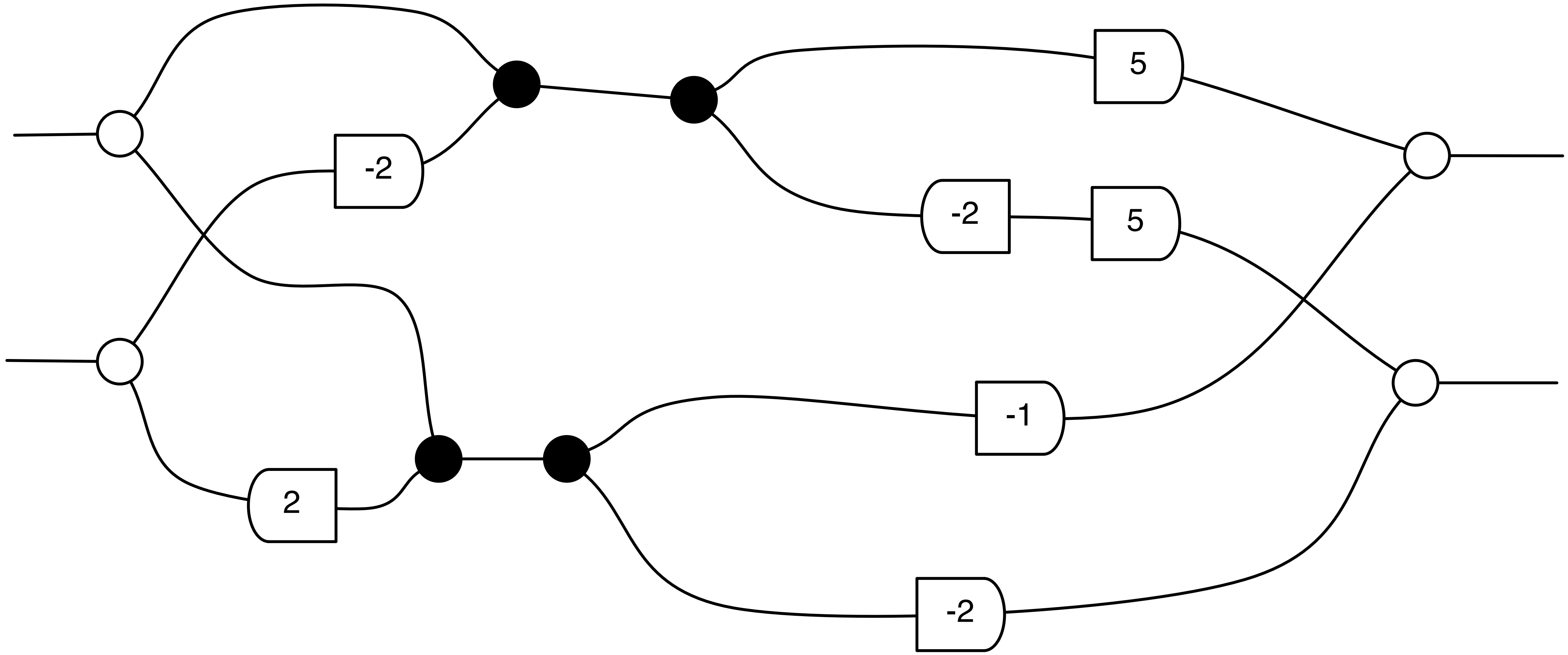

2
5
5
2
-1
-1



-2
5
5
2
-1
-1



5
-2
-2
5
2
-1
-2



$$\begin{pmatrix} 5 & -1 \\ -\frac{5}{2} & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & 2 \end{pmatrix}^{-1} = \frac{1}{5} \begin{pmatrix} 19 & -12 \\ -12 & 1 \end{pmatrix}$$

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